Numerical Partial Differential Equations:

Conservation Laws and Finite Volume Methods

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Outline

Burger's Equation in Primitive Form

Consider the problem:

$$\frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} = 0, \qquad x \in [0,1], \quad t > 0$$

$$u(x,0) = \begin{cases} 1 & x \leq 0.5 \\ 0 & x > 0.5 \end{cases} \qquad x \in [0,1],$$

$$u(0,t) = 1 \qquad t \geq 0$$

Consider the scheme:

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} + v_j^n \frac{v_j^n - v_{j-1}^n}{\Delta x} = 0 \implies v_i^{n+1} = v_i^n - \frac{\Delta t}{\Delta x} v_j^n (v_j^n - v_{j-1}^n)$$

The scheme doesn't even work for the first step $n = 0$ since

$$v_j^n(v_j^n - v_{j-1}^n) = \begin{cases} 1 \cdot (1-1) = 0, & \text{if } x_i \leq 0.5 \\ 0 \cdot (0-1) = 0, & \text{if } x_i > 0.5 \end{cases} \implies v_i^{n+1} = v_i^n$$

The numerical solution does not move at all.

Burger's Equation in Conservative Form

Consider the same problem in conservation form:

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2(x,t)}{2}\right) = 0, \qquad x \in [0,1], \quad t > 0$$

$$u(x,0) = \begin{cases} 1 & x \leq 0.5 \\ 0 & x > 0.5 \end{cases} \qquad x \in [0,1],$$

$$u(0,t) = 1 \qquad t \geq 0$$

Consider the scheme:

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} + \frac{1}{2}\frac{(v_j^n)^2 - (v_{j-1}^n)^2}{\Delta x} = 0 \implies v_i^{n+1} = v_i^n - \frac{\Delta t}{2\Delta x}((v_j^n)^2 - (v_{j-1}^n)^2)$$

The scheme does work since

$$(v_i^n)^2 - (v_{i-1}^n)^2 = \begin{cases} 1^2 - 1^2 = 0, & \text{if } v_i^n = v_{i-1}^n \\ 0^2 - 1^2 = -1, & \text{if } v_i^n \neq v_{i-1}^n \end{cases}$$

Wave speed is modified: An alternative way to look at the scheme

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} + \frac{1}{2}\frac{(v_j^n)^2 - (v_{j-1}^n)^2}{\Delta x} = 0 = \frac{v_i^{n+1} - v_i^n}{\Delta t} + \left(\frac{v_j^n + v_{j-1}^n}{2}\right)\frac{v_j^n - v_{j-1}^n}{\Delta x}$$

Conserved Scheme

PDE level:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad \text{(primitivity)} \qquad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{u^2}{2}) = 0 \quad \text{(conservative)}$$

Notice

$$\frac{d}{dt}\int_0^1 u\,dx = \int_0^1 \frac{\partial u}{\partial t}dx = -\int_0^1 \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)dx = -\frac{u^2(1,t)}{2} + \frac{u^2(0,t)}{2}$$

Numerical level:

$$\forall j = 1,2,...,N \quad \frac{dv_j(t)}{dt} = -v_j(t)\frac{v_j(t) - v_{j-1}(t)}{\Delta x}, \quad \text{(primitive)}$$

$$\frac{d}{dt}\sum_{j=1}^{N} v_j \Delta x = \frac{v_0^2}{2} - \frac{v_N^2}{2} - \frac{1}{2}\sum_{j=0}^{N}(v_j - v_{j-1})^2 \quad \text{(not conserved)}$$

$$\forall j = 1,2,...,N \quad \frac{dv_j(t)}{dt} = -\frac{v_j^2(t) - v_{j-1}^2(t)}{2\Delta x}, \quad \text{(conservative)}$$

$$\sum_{j=0}^{N}\frac{dv_j}{dt}\Delta x = \frac{v_0^2}{2} - \frac{v_N^2}{2} \quad \text{(conserved)}$$

Conservation Law and Finite Volume Method

Consider $u = u(x,t)$ satisfying the one dimension conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad x \in [0,1], \quad t \geq 0, \quad f(u)\text{: flux function}$$

Examples:

$$f(u) = a(x,t)u, \qquad \frac{\partial u}{\partial t} + \frac{\partial (au)}{\partial x} = \frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} + \frac{\partial a(x,t)}{\partial x}u = 0$$

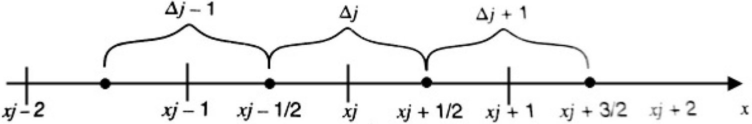
$$f(u) = u^2/2, \qquad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

Let a and b be two reals and $a < b \in [0,1]$. Notice that

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = -\int_a^b \frac{\partial f(u)}{\partial x}dx = -f(u(b,t)) + f(u(a,t))$$

indicating that u satisfies the conservation property.

FVM: $\frac{d}{dt}\int_{\Delta_j} u dx = -f(u,t)|_{x_{j+\frac{1}{2}}} + f(u,t)|_{x_{j-\frac{1}{2}}}$



cell end points: $0 = x_0 < x_{j-\frac{1}{2}} < x_N = 1 \quad j = 1, 2, ..., N$

cell and width:

$$\Delta_0 = [x_0, x_{\frac{1}{2}}], \quad \Delta_j = [x_{j-1/2}, x_{j+1/2}]_{j=1}^{N-1}, \quad \Delta_N = [x_{N-\frac{1}{2}}, x_N], \quad h_j = ||\Delta_j||$$

evaluation points and field values:

$$x_0 = 0,\ x_j = \frac{x_{j+\frac{1}{2}} + x_{j-\frac{1}{2}}}{2},\ x_N = 1, \quad v_j(t) \approx u(x_j, t), \quad f_j(t) = f(v_j, t) \approx f(u(x_j, t))$$

$$\frac{dv_j}{dt} h_j = -\frac{f_{j+1} + f_j}{2} + \frac{f_j + f_{j-1}}{2}, \quad \Longrightarrow \frac{dv_j}{dt} = -\frac{f_{j+1} - f_{j-1}}{2h_j}, \quad j = 1, ..., N-1$$

$$\frac{dv_0}{dt} h_0 = -\frac{f_1 + f_0}{2} + f_0, \quad \Longrightarrow \frac{dv_0}{dt} = -\frac{f_1 - f_0}{2h_0}, \quad \text{left cell}$$

$$\frac{dv_N}{dt} h_N = -f_N + \frac{f_{N-1} + f_N}{2}, \quad \Longrightarrow \frac{dv_N}{dt} = -\frac{f_N - f_{N-1}}{2h_N} \quad \text{right cell}$$

FVM: Conservative in the Discrete Sense

$$\frac{dv_0}{dt} = -\frac{f_1 - f_0}{2h_0} - \frac{\tau_-}{h_0}(f_0 - f_-), \qquad \text{left cell}$$

$$\frac{dv_j}{dt} = -\frac{f_{j+1} - f_{j-1}}{2h_j}, \qquad j = 1, ..., N-1$$

$$\frac{dv_N}{dt} = -\frac{f_N - f_{N-1}}{2h_N} + \frac{\tau_+}{h_N}(f_N - f_+) \qquad \text{right cell}$$

$\tau_\pm$ are called penalty parameters, used for imposing boundary conditions ($\tau_\pm = 1$). The scheme is conservative in the following sense: $\tau_\pm = 0$

$$\frac{d}{dt}\sum_{j=0}^{N} h_j v_j = \sum_{j=0}^{N} \frac{dv_j}{dt} h_j = -\frac{f_1 - f_0}{2} - \sum_{j=1}^{N-1} \frac{f_{j+1} - f_{j-1}}{2} - \frac{f_{N-1} - f_N}{2}$$

$$= -\frac{f_1 - f_0}{2} + \frac{f_0 + f_1}{2} - \frac{f_N + f_{N-1}}{2} - \frac{f_N - f_{N-1}}{2} = f_0 - f_N$$

mimicking

$$\frac{d}{dt}\int_0^1 u(x,t)\,dx = -f(u(1,t)) + f(u(0,t))$$

The scheme remains conserved when BCs are enforced.

Stability

Let $f(u) = a(x)u(x,t)$, $a(x) > 0$. Consider the problem:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, && x \in [0,1], \quad t \geq 0, \\
&u(x,0) = u_0(x), && x \in [0,1] \\
&u(0,t) = g(t), && t \geq 0
\end{aligned}
$$

FVM scheme:

$$
\begin{aligned}
&\frac{dv_0}{dt} = -\frac{f_1 - f_0}{2h_0} - \frac{\tau_-}{h_0}(f_0 - f_-), && \text{left cell}, \tau_- = 1 \\
&\frac{dv_j}{dt} = -\frac{f_{j+1} - f_{j-1}}{2h_j}, && j = 1, ..., N-1 \\
&\frac{dv_N}{dt} = -\frac{f_N - f_{N-1}}{2h_N} && \text{right cell} \\
&v_j(0) = u_0(x_j) && j = 0, 1, ..., N
\end{aligned}
$$

Stability: $a_j v_j = f_j$, Find $\sum_{j=0}^{N} a_j h_j v_j^2$

$$\frac{dv_0}{dt} = -\frac{f_1 - f_0}{2h_0} - \frac{1}{h_0}(f_0 - f_-), \quad \underbrace{\frac{dv_j}{dt} = -\frac{f_{j+1} - f_{j-1}}{2h_j}}_{j=1,2,\dots,N-1}, \quad \frac{dv_N}{dt} = -\frac{f_N - f_{N-1}}{2h_N}$$

$$2v_0 a_0 h_0 \frac{dv_0}{dt} = -f_0(f_1 - f_0) - 2f_0(f_0 - f_-) = -f_0 f_1 - f_0^2 + 2f_0 f_-$$

$$\sum_{j=1}^{N-1} 2v_j a_j h_j \frac{dv_j}{dt} = -\sum_{j-1}^{N-1} f_j(f_{j+1} - f_{j-1}) = f_1 f_0 - f_{N-1} f_N,$$

$$2v_N a_N h_N \frac{dv_N}{dt} = f_N f_{N-1} - f_N^2$$

$$\frac{d}{dt}\sum_{j=0}^{N} a_j h_j v_j^2(t) = -f_0^2 + 2f_0 f_- - f_N^2 = -f_N^2 - (f_0 - f_-(t))^2 + f_-^2(t) \le f_-^2(t)$$

$$\implies \sum_{j=0}^{N} a_j h_j v_j^2(t) \le \sum_{j=0}^{N} a_j h_j v_j^2(0) + \int_0^t f_-^2(t')dt' = M \implies \sum_{j=0}^{N} h_j v_j^2(t) < cM$$

Mutidimensional Problem

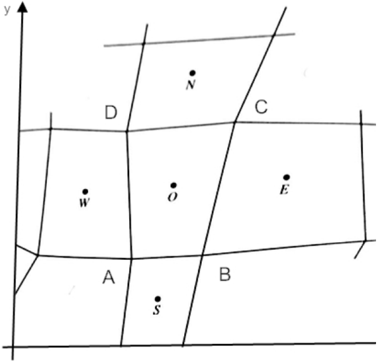
Consider $u = u(x, y, t)$ satisfying

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0$$

$$F = F(x, y, t, u), \quad G = G(x, y, t, u)$$

Δ: finite volume
$\partial\Delta$ volume boundary.

$$\frac{d}{dt}\int_{\Delta} u\, dxdy = -\int_{\partial\Delta} F(x, y, t, u)dy + \int_{\partial\Delta} G(x, y, t, u)dx$$



Mutidimensional Problem

Δ: Quad ABCD
$\partial\Delta$ volume boundary.

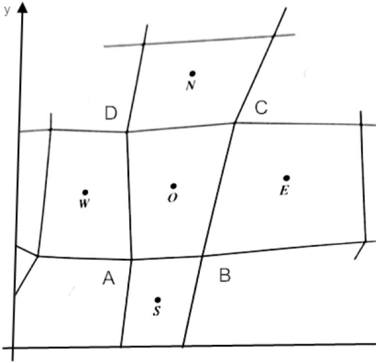
$$\frac{d}{dt}\int_{\Delta} u\,dxdy = -\int_{\partial\Delta} F dy + \int_{\partial\Delta} G dx$$

Left hand side term

$$\frac{d}{dt}\int_{\Delta} u\,dxdy \approx ||\Delta||\frac{dv_o}{dt}$$

boundary integral term

$$\int_{\partial\Delta} F dy \approx (y_B - y_A)\frac{F_O + F_S}{2} + (y_C - y_B)\frac{F_O + F_E}{2} + (y_D - y_C)\frac{F_O + F_N}{2} + (y_A - y_D)\frac{F_O + F_W}{2}$$



boundary integral term

$$\int_{\partial\Delta} G dy \approx (x_B - x_A)\frac{G_O + G_S}{2} + (x_C - x_B)\frac{G_O + G_E}{2} + (x_D - x_C)\frac{G_O + G_N}{2} + (x_A - x_D)\frac{G_O + G_W}{2}$$

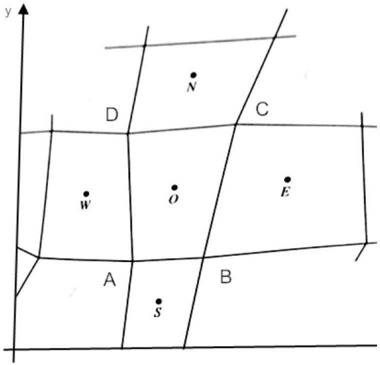
Mutidimensional Problem

Δ: Quad ABCD
$\partial\Delta$ volume boundary.

$$\frac{d}{dt}\int_{\Delta} u\,dxdy = -\int_{\partial\Delta} Fdy + \int_{\partial\Delta} Gdx$$

FV scheme:

$$\begin{aligned}||\Delta||\frac{dv_O}{dt} &= (y_B - y_A)\frac{F_O + F_S}{2} + (y_C - y_B)\frac{F_O + F_E}{2} \\ &+ (y_D - y_C)\frac{F_O + F_N}{2} + (y_A - y_D)\frac{F_O + F_W}{2} \\ &- (G_B - G_A)\frac{G_O + G_S}{2} - (x_C - x_B)\frac{G_O + G_E}{2} \\ &- (x_D - x_C)\frac{G_O + G_N}{2} - (x_A - x_D)\frac{G_O + G_W}{2}\end{aligned}$$



Summing over all the cells we have all the F and G terms canceled excepts for the boundary terms, establishing the conservative property in the discrete sense.

Summary

- conservative property preserved
- can be generalized to multidimensional problem easily
- can be adopted to unstructured meshes
  - 2D: triangular, quadrilateral, and polygonal cells
  - 3D: tetrahedral and hexahedral cells
- suitable for problems with periodicity
  - pdes on spherical surfaces